Harmonic Oscillator Wavefunctions

\[ \Psi_v = N_v H_v(y)e^{-y^2/2\alpha^2} \]

where: \( y = x/\alpha, \ \alpha = (\hbar^2/mk)^{1/4}, \ N_v = (1/\pi\alpha^2)^{1/4}(1/2^v v!)^{1/2} \)

Finding the Expectation Value of \( x^2 \)

\[ \langle x^2 \rangle = \int \Psi_v x^2 \Psi_v d\tau = N_v^2 \int H_v e^{-y^2/2\alpha^2} y^2 H_v e^{-y^2/2\alpha^2} dx \]

\[ = \alpha^3 N_v^2 \int H_v y^2 H_v e^{y^2/2\alpha^2} dy \]

Use Recursion Relation: \( yH_v = vH_{v-1} + (1/2)H_{v+1} \)

\[ = \alpha^3 N_v^2 \int H_v [vH_{v-1} + (1/2)H_{v+1}] e^{-y^2/2\alpha^2} dy \]

Use Recursion Relation Again.

Use \( yH_{v+1} \) & \( yH_{v-1} \) in place of \( yH_v \).

\[ = \alpha^3 N_v^2 \int H_v [v(v-1)H_{v-2} + v(1/2)H_v + (1/2)(v+1)H_{v+1} e^{-y^2/2\alpha^2} dy \]

Integrals with \( v' \neq v \) are equal to zero.

Integrals with \( v' = v \) are equal to \( \pi^{1/2}2^v v! \).

\[ = \alpha^3 N_v^2 \int H_v [v(1/2)H_v + (1/2)(v+1)H_{v+1}] e^{-y^2/2\alpha^2} dy \]

\[ = \alpha^3 N_v^2 \left[ v(1/2)\pi^{1/2}2^v v! + (1/2)(v+1)\pi^{1/2}2^v v! \right] \]

\[ = (\hbar^2/mk)^{1/2}(1/\pi\alpha^2)^{1/2}(1/2^v v!)[v(1/2)\pi^{1/2}2^v v! + (1/2)(v+1)\pi^{1/2}2^v v!] \]

\[ = (\hbar^2/mk)^{1/2}(v + 1/2) \]

\[ = (v + 1/2)\hbar/(mk)^{1/2} \]

\[ \rightarrow \text{Amplitude of swing increases w increasing quantum number.} \]

Finding the Expectation Value of the Potential Energy

\[ \langle V \rangle = 1/2k\langle x^2 \rangle = 1/2k(v + 1/2)\hbar/(mk)^{1/2} \]

\[ = (1/2)(v + 1/2)\hbar(k/m)^{1/2} \]

\[ = (1/2)(v + 1/2)\hbar \omega \]

\[ \langle V \rangle = 1/2E_v \]

\[ \rightarrow E_{kin} = 1/2E_v = E_{pot} \]

The Virial Theorem: If the potential energy has the form \( V=ax^b \) then the mean potential and kinetic energies are related by \( 2\langle E_k \rangle = b\langle V \rangle \)